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Effects due to body-forces and body-couples in the interior of a micropolar elastic half-space

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SUMMARY
A general solution of the equilibrium equations is obtained for a half-space with a stress-free boundary and arbitrary but axisymmetric distribution of body-forces and body-couples in the interior of the half-space. Few particular cases have been investigated in detail. The stresses, displacements and rotation have been obtained at the boundary of the half-space. Numerical results have been displayed graphically to illustrate the micropolar effects.

1. Introduction

The asymmetric theory of elasticity initiated by Voigt [1] in 1887 and further developed by E. Cosserat and F. Cosserat [2] in 1909 has gained a renewed momentum during the recent years. The linear micropolar theory has been given by Kuvchinski and Aero [3], Palmov [4] and Eringen and Suhubi [5].

The axisymmetric Lamb's problem in a semi-infinite micropolar elastic solid has been solved recently by Nowacki [6]. Puri [7] and Dhaliwal [8] have obtained solutions respectively for stress concentration and thermoelastic problems for a semi-infinite micropolar elastic solid. The solution to the axisymmetric Boussinesq problem has been obtained by Dhaliwal [9] recently.

In this paper we have considered the boundary $z = 0$ of the half-space $z \geqq 0$ to be stress-free. The case when the boundary of the half-space is fixed has been considered by the authors [10] separately. In an elastic half-space a general solution of the equations of equilibrium (c), expressed in cylindrical coordinates, has been derived for an arbitrary distribution of axially-symmetric body-forces and body-couples. The method employs the technique of integral transforms. By demanding that the Laplace transforms of displacements and rotation have no singularities in the right-half plane of the transform variable, we have obtained four linear algebraic equations involving three Hankel-transformed components of displacements and rotation. Only three of these equations are independent thus showing the independence of the displacement and the rotation fields, as expected. The following four cases have been considered in detail: (i) conservative body-forces (ii) body-couple as the curl of a vector function (iii) concentrated body-force (iv) concentrated body-couple.

The corresponding classical results have been obtained by letting the micropolar constant α tend to zero. The stresses, couple-stresses, displacements and rotation, have been obtained at the boundary $z = 0$ of the micropolar half-space. Numerical results for the stresses, displacements and rotation have been displayed graphically.

2. General solution of the basic equations

We consider the system of cylindrical coordinates r, φ, z and assume the axial-symmetry. Hence, we shall be concerned with the following components of the vectors:

$$\boldsymbol{u} \equiv (u_r, 0, u_z),\ \ \boldsymbol{w} \equiv (0, w_\varphi, 0),\ \ \boldsymbol{X} = (X_r, 0, X_z),\ \ \boldsymbol{Y} \equiv (0, Y_\varphi, 0),$$

where $\boldsymbol{u}$ denotes the macro-displacement vector, $\boldsymbol{w}$ is the micro-rotation vector and the body-force vector and the body-couple vector are respectively given by $\boldsymbol{X}$ and $\boldsymbol{Y}$.

As shown by Nowacki [6] the equations of equilibrium of asymmetric elasticity can be decomposed into two mutually independent sets of equations in the case of axisymmetry. In what follows, we shall be concerned with the set of equations:

$$\begin{aligned}
&(\mu + \alpha)\left(\nabla^2 - \frac{1}{r^2}\right)u_r + (\lambda + \mu - \alpha)\frac{\partial e}{\partial r} - 2\alpha\frac{\partial w_\varphi}{\partial z} + \rho X_r = 0,\\
&(\mu + \alpha)\nabla^2 u_z + (\lambda + \mu - \alpha)\frac{\partial e}{\partial r} + 2\alpha\cdot\frac{1}{r}\frac{\partial}{\partial r}(rw_\varphi) + \rho X_z = 0,\\
&(\gamma + \varepsilon)\left(\nabla^2 - \frac{1}{r^2}\right)w_\varphi - 4\alpha w_\varphi + 2\alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + JY_\varphi = 0,
\end{aligned} \tag{1}$$

in which λ, μ, α, γ, ε are the elastic constants of the micropolar material, ρ is the density, J is the rotational inertia, and

$$e = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z},\quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{2}$$

Corresponding to the displacement vector $\boldsymbol{u} = (u_r, 0, u_z)$ and to the rotation vector $\boldsymbol{w} = (0, w_\varphi, 0)$, the components of the stress tensors are given by

$$\begin{aligned}
&(\sigma_{rr}, \sigma_{\varphi\varphi}, \sigma_{zz})(r, z) = 2\mu\left(\frac{\partial u_r}{\partial r}, \frac{u_r}{r}, \frac{\partial u_z}{\partial z}\right) + \lambda(e, e, e),\\
&(\sigma_{rz}, \sigma_{zr})(r, z) = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \mp \alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \pm 2\alpha w_\varphi,\\
&(\mu_{r\varphi}, \mu_{\varphi r})(r, z) = \gamma\left(\frac{\partial w_\varphi}{\partial r} - \frac{w_\varphi}{r}\right) \pm \varepsilon\left(\frac{\partial w_\varphi}{\partial r} + \frac{w_\varphi}{r}\right),\\
&(\mu_{\varphi z}, \mu_{z\varphi})(r, z) = (\gamma \mp \varepsilon)\frac{\partial w_\varphi}{\partial z}.
\end{aligned} \tag{3}$$

Let $(\tilde{u}_r, \tilde{w}_\varphi, \tilde{X}_r, \tilde{Y}_\varphi, \tilde{\sigma}_{zr}, \tilde{\mu}_{z\varphi})(\xi, z)$ and $(\tilde{u}_z, \tilde{X}_z, \tilde{\sigma}_{zz})(\xi, z)$ denote the Hankel-transforms respectively of order 1 and 0 of the corresponding components $(u_r, w_\phi, X_r, Y_\phi, \sigma_{zr}, \mu_{z\varphi})(r, z)$ and $(u_z, X_z, \sigma_{zz})(r, z)$. We assume that all stresses, displacements and rotations vanish at infinity. So, Hankel-transformation of the equations (3) yields

$$
\begin{aligned}
&\tilde{\sigma}_{zz}(\xi, z) = (\lambda + 2\mu)D\tilde{u}_z + \lambda\xi\tilde{u}_r,\\
&(\tilde{\sigma}_{rz}, \tilde{\sigma}_{zr})(\xi, z) = (\mu \mp \alpha)D\tilde{u}_r - (\mu \pm \alpha)\xi\tilde{u}_z \pm 2\alpha\tilde{w}_\varphi,\\
&(\tilde{\mu}_{\varphi z}, \tilde{\mu}_{z\varphi})(\xi, z) = (\gamma \mp \varepsilon)D\tilde{w}_\varphi,
\end{aligned} \tag{4}
$$

where operator D is given by $D = d/dz$. If we define the Laplace-transform of the function $f(\xi, z)$ as $\bar{f}(\xi, p)$, use the notation

$$
\left(\frac{d^n f}{dz^n}\right)_{z=0} = f^{(n)}, \qquad n = 0, 1,
$$

and take the Hankel and Laplace-transform of the system of equations (1), we obtain the algebraic system of equations:

$$
\begin{aligned}
&[(\mu + \alpha)p^2 - (\lambda + 2\mu)\xi^2]\bar{\tilde{u}}_r - (\lambda + \mu - \alpha)\xi p\bar{\tilde{u}}_z - 2\alpha p\bar{\tilde{w}}_\varphi = A(\xi, p),\\
&(\lambda + \mu - \alpha)\xi p\bar{\tilde{u}}_r + [(\lambda + 2\mu)p^2 - (\mu + \alpha)\xi^2]\bar{\tilde{u}}_z + 2\alpha\xi\bar{\tilde{w}}_\varphi = B(\xi, p),\\
&2\alpha p\bar{\tilde{u}}_r + 2\alpha\xi\bar{\tilde{u}}_z + [(\gamma + \varepsilon)(p^2 - \xi^2) - 4\alpha]\bar{\tilde{w}}_\varphi = C(\xi, p),
\end{aligned} \tag{5}
$$

where we have assumed that the displacements and rotation vanish at infinity and the right-hand sides of these equations are given by

$$
\begin{aligned}
&A(\xi, p) = (\mu + \alpha)(\tilde{u}_r^{(1)} + p\tilde{u}_r^{(0)}) - (\lambda + \mu - \alpha)\xi\tilde{u}_z^{(0)} - 2\alpha\tilde{w}_\varphi^{(0)} - \rho\bar{\tilde{X}}_r,\\
&B(\xi, p) = (\lambda + \mu - \alpha)\xi\tilde{u}_r^{(0)} + (\lambda + 2\mu)(\tilde{u}_z^{(1)} + p\tilde{u}_z^{(0)}) - \rho\bar{\tilde{X}}_z,\\
&C(\xi, p) = 2\alpha\tilde{u}_r^{(0)} + (\gamma + \varepsilon)(\tilde{w}_\varphi^{(1)} + p\tilde{w}_\varphi^{(0)}) - J\bar{\tilde{Y}}_\varphi.
\end{aligned} \tag{6}
$$

Solving the system of equations (5), we obtain the transformed components of the displacement and rotation vectors:

$$
(\lambda + 2\mu)(\mu + \alpha)(\gamma + \varepsilon)(\bar{\tilde{u}}_r, \bar{\tilde{u}}_z, \bar{\tilde{w}}_\varphi)(\xi, p) = (D_1, D_2, D_3)/D, \tag{7}
$$

where D_1, D_2, D_3 and D are given by

$$
\begin{aligned}
&D_1(\xi, p) = \Phi(p^2, \xi^2)A + \Psi(p, \xi)B + \Theta(p, \xi)C,\\
&D_2(\xi, p) = -\Psi(p, \xi)A + \Phi(-\xi^2, -p^2)B - \Theta(-\xi, -p)C,\\
&D_3(\xi, p) = (\lambda + 2\mu)(p^2 - \xi^2)[(\mu + \alpha)(p^2 - \xi^2)C - 2\alpha(pA + \xi B)],\\
&D(\xi, p) = (p^2 - \xi^2)^2(p^2 - \zeta^2),
\end{aligned} \tag{8}
$$

in which the functions Φ, Ψ and Θ are expressed as

$$
\begin{aligned}
&\Phi(p^2, \xi^2) = (\gamma + \varepsilon)[(\lambda + 2\mu)p^4 - (\lambda + 3\mu - \alpha)p^2\xi^2 + (\mu + \alpha)\xi^4] + 4\alpha\mu\xi^2 - 4\alpha(\lambda + 2\mu)p^2,\\
&\Psi(p, \xi) = p\xi[(\lambda + \mu - \alpha)(\gamma + \varepsilon)(p^2 - \xi^2) - 4\alpha(\lambda + \mu)],\\
&\Theta(p, \xi) = 2\alpha(\lambda + 2\mu)(p^2 - \xi^2)p,
\end{aligned} \tag{9}
$$

with ζ and m^2 defined as

$$
\zeta = \sqrt{\xi^2 + m^2}, \quad m^2 = \frac{4\alpha\mu}{(\mu + \alpha)(\gamma + \varepsilon)}. \tag{10}
$$

The points of singularities in (7) are a single pole at $p = \pm\zeta$, and a double pole at $p = \pm\xi$. By demanding that the Laplace transforms of displacement and rotation components have no singularities in the right half-plane of the transform variable p, we obtain the following nine conditions:

$$D_j|_{p=\zeta} = D_j|_{p=\xi} = \left.\frac{\partial D_j}{\partial p}\right|_{p=\xi} = 0, \quad j = 1, 2, 3, \tag{11}$$

which yields only the following four distinct relations:

$$\begin{aligned}
&A_1 + B_1 = 0,\\
&\Delta_1(\xi)A_1 + \Delta_2(\xi)B_1 + 2\alpha(\lambda + 2\mu)\xi C_1 - 2\alpha(\lambda + \mu)\xi\left[\left(\frac{\partial A}{\partial p}\right)_1 + \left(\frac{\partial B}{\partial p}\right)_1\right] = 0,\\
&-\Delta_2(\xi)A_1 - \Delta_3(\xi)B_1 - 2\alpha(\lambda + 2\mu)\xi C_1 + 2\alpha(\lambda + \mu)\xi\left[\left(\frac{\partial A}{\partial p}\right)_1 + \left(\frac{\partial B}{\partial p}\right)_1\right] = 0,\\
&k^2(\zeta A_2 + \xi B_2) - m^2 C_2 = 0,
\end{aligned} \tag{12}$$

in which we have used the following notations:

$$\begin{aligned}
&(A_1, B_1, C_1) = (A, B, C)_{p=\xi},\\
&(A_2, B_2, C_2) = (A, B, C)_{p=\zeta},\\
&\left(\frac{\partial A}{\partial p}\right)_1 = \left(\frac{\partial A}{\partial p}\right)_{p=\xi} = (\mu + \alpha)\tilde{u}_r^{(0)} - \rho\left(\frac{\partial \bar{\bar{X}}_r}{\partial p}\right)_1,\\
&\left(\frac{\partial B}{\partial p}\right)_1 = \left(\frac{\partial B}{\partial p}\right)_{p=\xi} = (\lambda + 2\mu)\tilde{u}_z^{(0)} - \rho\left(\frac{\partial \bar{\bar{X}}_z}{\partial p}\right)_1,\\
&\left(\frac{\partial C}{\partial p}\right)_1 = \left(\frac{\partial C}{\partial p}\right)_{p=\xi} = (\gamma + \varepsilon)\tilde{w}_\varphi^{(0)} - J\left(\frac{\partial \bar{\bar{Y}}_\varphi}{\partial p}\right)_1,\\
&\left(\left(\frac{\partial \bar{\bar{X}}_r}{\partial p}\right)_1, \left(\frac{\partial \bar{\bar{X}}_z}{\partial p}\right)_1, \left(\frac{\partial \bar{\bar{Y}}_\varphi}{\partial p}\right)_1\right) = \left(\frac{\partial \bar{\bar{X}}_r}{\partial p}, \frac{\partial \bar{\bar{X}}_z}{\partial p}, \frac{\partial \bar{\bar{Y}}_\varphi}{\partial p}\right)_{p=\xi},
\end{aligned} \tag{13}$$

and Δ_1, Δ_2, Δ_3 are given by

$$\begin{aligned}
&\Delta_1(\xi) = (\lambda + \mu - \alpha)(\gamma + \varepsilon)\xi^2 - 4\alpha(\lambda + 2\mu),\\
&\Delta_2(\xi) = (\lambda + \mu - \alpha)(\gamma + \varepsilon)\xi^2 - 2\alpha(\lambda + \mu),\\
&\Delta_3(\xi) = (\lambda + \mu - \alpha)(\gamma + \varepsilon)\xi^2 + 4\alpha\mu.
\end{aligned} \tag{14}$$

The micropolar constant α is contained in k^2 which is defined as

$$k^2 = \frac{2\alpha}{\mu + \alpha}.$$

The system of equations (12) is linearly dependent as the sum of the second and the third equation yields the first equation. However, its solution depends on the prescribed conditions at the boundary $z = 0$ of the half-space $z \geqq 0$. In this paper we shall consider the case of a free boundary.

3. General solution for stress-free boundary

Since the boundary $z = 0$ is stress-free, we have

$$\sigma_{zz}(r, 0) \equiv 0, \quad \sigma_{zr}(r, 0) \equiv 0, \quad \mu_{z\varphi}(r, 0) \equiv 0. \tag{15}$$

Using these boundary conditions in the relations (4) given earlier, we obtain

$$\begin{aligned}
&(\lambda + 2\mu)\tilde{u}_z^{(1)}(\xi, 0) = -\lambda\xi\tilde{u}_r^{(0)},\\
&(\mu + \alpha)\tilde{u}_r^{(1)}(\xi, 0) = (\mu - \alpha)\xi\tilde{u}_z^{(0)} + 2\alpha\tilde{w}_\varphi^{(0)},\\
&\tilde{w}_\varphi^{(1)}(\xi, 0) \equiv 0.
\end{aligned} \tag{16}$$

The last equation implies that the couple-stress components $\mu_{\varphi z}$ and $\mu_{z\varphi}$ vanish identically at the boundary.

Using these relations in (6) and the utilizing (12), we obtain the following system of equations:

$$\begin{aligned}
&\tilde{u}_r^{(0)} + \tilde{u}_z^{(0)} = M_1,\\
&(\mu R - \alpha F_1)\tilde{u}_r^{(0)} + (\mu R - \alpha F_2)\tilde{u}_z^{(0)} + 2\alpha\mu l^2(\lambda + 2\mu)\xi\tilde{w}_\varphi^{(0)} = \frac{1}{2\xi}M_2,\\
&[\mu R - \alpha(\lambda\mu - \mu^2 - \lambda\alpha - \mu\alpha)]\tilde{u}_r^{(0)} + [\mu R + \alpha(\lambda^2 + 4\mu^2 + 3\lambda\mu)]\tilde{u}_z^{(0)}\\
&\qquad\qquad + 2\alpha\mu l^2(\lambda + 2\mu)\xi\tilde{w}_\varphi^{(0)} = \frac{1}{2\xi}M_2^*,\\
&\xi^2\tilde{u}_r^{(0)} + \zeta\xi\tilde{u}_z^{(0)} - \zeta\tilde{w}_\varphi^{(0)} = \frac{1}{2\mu}M_3,
\end{aligned} \tag{17}$$

in which the quantities involved are given by

$$\begin{aligned}
&R = 2\mu l^2(\lambda + \mu - \alpha)\xi^2, \quad F_1 = 3\lambda\mu + 5\mu^2 - \alpha(\lambda + \mu), \quad F_2 = (\lambda + 2\mu)(\mu - \alpha)\\
&M_1 = \rho[(X_r)_1 + (X_z)_1], \quad l^2 = \frac{\gamma + \varepsilon}{2\mu}\\
&M_3 = \rho[\zeta(X_r)_2 + \xi(X_z)_2] - \frac{J}{l^2}(Y_\varphi)_2\\
&M_2 = \rho[\Delta_1(\bar{\bar{X}}_r)_1 + \Delta_2(\bar{\bar{X}}_z)_1] + 2\alpha\xi\left\{(\lambda + 2\mu)J(\bar{\bar{Y}}_\varphi)_1 + (\lambda + \mu)\left(\left(\frac{\partial A}{\partial p}\right)_1 + \left(\frac{\partial B}{\partial p}\right)_1\right)\right\},\\
&M_2^* = \rho[\Delta_2(\bar{\bar{X}}_r)_1 + \Delta_3(\bar{\bar{X}}_z)_1] + 2\alpha\xi\left\{(\lambda + 2\mu)J(\bar{\bar{Y}}_\varphi)_1 + (\lambda + \mu)\left(\left(\frac{\partial A}{\partial p}\right)_1 + \left(\frac{\partial B}{\partial p}\right)_1\right)\right\}.
\end{aligned} \tag{18}$$

We find that the above system of equations (17) is linearly dependent as the difference of the second and the third equations yields the first equation. Hence, the system to be solved for $\tilde{u}_r^{(0)}$, $\tilde{u}_z^{(0)}$ and $\tilde{w}_\varphi^{(0)}$ may consist of the first, second and the fourth equation whose solution will give us

$$\tilde{u}_z^{(0)}(\xi, 0) = \frac{\overline{M}_4(\xi)}{\overline{Y}(\xi)}, \quad \tilde{u}_r^{(0)}(\xi, 0) = \frac{\overline{M}_5(\xi)}{\overline{Y}(\xi)}, \quad \tilde{w}_\varphi^{(0)}(\xi, 0) = \frac{\overline{M}_6(\xi)}{\overline{Y}(\xi)}, \tag{19}$$

in which $\overline{M}_4$, $\overline{M}_5$, $\overline{M}_6$, $\overline{M}_2$ and $\overline{Y}$ are given by

$$\overline{M}_4(\xi) = \left[(2\lambda + 3\mu)\zeta - \frac{\mu}{\alpha}(\lambda + \mu - \alpha)l^2\xi^2\zeta - (\lambda + 2\mu)l^2\xi^3\right]$$
$$\times \frac{M_1}{\xi} + \frac{\zeta\overline{M}_2}{2\alpha\xi} + (\lambda + 2\mu)l^2\xi M_3,$$
$$\overline{M}_5(\xi) = \frac{\zeta}{\xi}\left[\left\{\frac{2(\lambda + \mu)}{m^2}\xi^2 - (\lambda + 2\mu)\right\} M_1 - \frac{\overline{M}_2}{2\alpha}\right] - (\lambda + 2\mu)l^2\xi M_3,$$
$$\overline{M}_6(\xi) = \left[(2\lambda + 3\mu)\zeta - \frac{\mu}{\alpha}(\lambda + \mu - \alpha)l^2\xi^2(\zeta - \xi) - (\lambda + 2\mu)\xi\right]$$
$$\times M_1 + (\zeta - \xi)\frac{\overline{M}_2}{2\alpha} - (\lambda + \mu)M_3,$$
$$\overline{M}_2 = \rho[\Delta_1(X_r)_1 + \Delta_2(X_z)_1] + 2\alpha\xi[(\lambda + 2\mu)J(Y_\varphi)_1 - (\lambda + \mu)\rho\left(\left(\frac{\partial\bar{\bar{X}}_r}{\partial p}\right)_1 + \left(\frac{\partial\bar{\bar{X}}_z}{\partial p}\right)_1\right)$$
$$\overline{Y}(\xi) = \left(\frac{2\mu^2}{1 - 2\nu}\right)\frac{1}{\bar{\Delta}(\xi)}, \tag{20}$$

where $\bar{\Delta}(\xi)$ is given by

$$\bar{\Delta}(\xi) = \frac{1}{\zeta + 2(1 - \nu)l^2\xi^2(\zeta - \xi)}.$$

M_1 and M_3 involved in these expressions are the same as given earlier in (18). So, the displacement and rotation components at the stress-free boundary, given by (19), can be rewritten as:

$$u_z(r, 0) = \frac{1 - 2\nu}{2\mu^2} H_0[\overline{M}_4(\xi)\bar{\Delta}(\xi);\ \xi \to r],$$
$$u_r(r, 0) = \frac{1 - 2\nu}{2\mu^2} H_1[\overline{M}_5(\xi)\bar{\Delta}(\xi);\ \xi \to r], \tag{21}$$
$$w_\varphi(r, 0) = \frac{1 - 2\nu}{2\mu^2} H_1[\overline{M}_6(\xi)\bar{\Delta}(\xi);\ \xi \to r],$$

in which the integral operator H_n is defined as

$$H_n[f(\xi);\ \xi \to r] = \int_0^\infty \xi f(\xi)J_n(\xi r)d\xi, \quad n = 0, 1. \tag{22}$$

Use of (21) in the stress-strain relations (3), yields the following non-vanishing stress components at the stress-free boundary.

$$\sigma_{rr}(r, 0) = \left(\frac{1 - 2\nu}{\mu}\right)\int_0^\infty \xi\overline{M}_5(\xi)\bar{\Delta}(\xi)\left[\frac{\xi}{1 - \nu}J_0(\xi r) - \frac{1}{r}J_1(\xi r)\right]d\xi,$$
$$\sigma_{\varphi\varphi}(r, 0) = \left(\frac{1 - 2\nu}{\mu}\right)\int_0^\infty \xi\overline{M}_5(\xi)\bar{\Delta}(\xi)\left[\frac{\nu}{1 - \nu}\xi J_0(\xi r) + \frac{1}{r}J_1(\xi r)\right]d\xi,$$

$$\sigma_{rz}(r, 0) = 2\mu k^2\left[w_\varphi(r,0) - \left(\frac{1-2\nu}{2\mu^2}\right)\int_0^\infty \xi^2 \overline{M}_4(\xi)\bar{\Delta}(\xi)J_1(\xi r)\,d\xi\right],$$

$$\mu_{r\varphi}(r, 0) = 2\mu l^2\left(\frac{\partial}{\partial r} - \frac{\gamma-\varepsilon}{2\mu l^2}\cdot\frac{1}{r}\right)w_\varphi(r, 0), \tag{23}$$

$$\mu_{\varphi r}(r, 0) = (\gamma-\varepsilon)\left(\frac{\partial}{\partial r} - \frac{2\mu l^2}{\gamma-\varepsilon}\cdot\frac{1}{r}\right)w_\varphi(r, 0),$$

$$\mu_{z\varphi}(r, 0) = \mu_{\varphi z}(r, 0) = 0.$$

Equations (21) and (23) give the general solution to the problem of stress-free boundary.

To derive the classical part from the general micropolar solution, we note that in the limiting case (i.e., $\alpha \to 0$) the expressions for $\overline{M}_4$, $\overline{M}_5$, $\overline{M}_6$ and $\bar{\Delta}$ given by (20) become:

$$\lim_{\alpha\to 0}[\overline{M}_4(\xi)] = -\mu\rho\bar{\bar{X}}_r(\xi,\xi) + (\lambda+2\mu)\rho\bar{\bar{X}}_z(\xi,\xi) - (\lambda+\mu)\xi\rho\frac{\partial}{\partial p}$$
$$\times[\bar{\bar{X}}_r(\xi,\xi) + \bar{\bar{X}}_z(\xi,\xi)],$$

$$\lim_{\alpha\to 0}[\overline{M}_5(\xi)] = (\lambda+2\mu)\rho\bar{\bar{X}}_r(\xi,\xi) - \mu\rho\bar{\bar{X}}_z(\xi,\xi) + (\lambda+\mu)\xi\rho\frac{\partial}{\partial p}$$
$$\times[\bar{\bar{X}}_r(\xi,\xi) + \bar{\bar{X}}_z(\xi,\xi)],$$

$$\lim_{\alpha\to 0}[\bar{\Delta}(\xi)] = \frac{1}{\xi},\quad \lim_{\alpha\to 0}[\overline{M}_6(\xi)] = 0 \tag{24}$$

Hence, we obtain the general solution for the corresponding classical theory problem:

$$u_z^c(r, 0) = -\frac{\rho}{2\mu}\int_0^\infty\Big[(1-2\nu)\bar{\bar{X}}_r(\xi,\xi) - 2(1-\nu)\bar{\bar{X}}_z(\xi,\xi)$$
$$+\xi\frac{\partial}{\partial p}\{\bar{\bar{X}}_r(\xi,\xi) + \bar{\bar{X}}_z(\xi,\xi)\}\Big]J_0(\xi r)\,d\xi,$$

$$u_r^c(r, 0) = \frac{\rho}{2\mu}\int_0^\infty\Big[2(-\nu)\bar{\bar{X}}_r(\xi,\xi) - (1-2\nu)\bar{\bar{X}}_z(\xi,\xi)$$
$$+\xi\frac{\partial}{\partial p}\{\bar{\bar{X}}_r(\xi,\xi) + \bar{\bar{X}}_z(\xi,\xi)\}\Big]J_1(\xi r)\,d\xi,$$

$$w_\varphi^c(r, 0) \equiv 0,$$

$$\sigma_{rr}^c(r, 0) = \rho\int_0^\infty\left[2(1-\nu)\bar{\bar{X}}_r(\xi,\xi) - (1-2\nu)\bar{\bar{X}}_z(\xi,\xi) + \xi\frac{\partial}{d\varrho}\{\bar{\bar{X}}_r(\xi,\xi) + \bar{\bar{X}}_z(\xi,\xi)\}\right]$$
$$\times\left[\frac{\xi}{1-\nu}J_0(\xi r) + \frac{1}{r}J_1(\xi r)\right]d\xi,$$

$$\sigma_{\varphi\varphi}^c(r, 0) = \rho\int_0^\infty\left[2(1-\nu)\bar{\bar{X}}_r(\xi,\xi) - (1-2\nu)\bar{\bar{X}}_z(\xi,\xi) + \xi\frac{\partial}{\partial p}\{\bar{\bar{X}}_r(\xi,\xi) + \bar{\bar{X}}_z(\xi,\xi)\}\right]$$
$$\times\left[\left(\frac{\nu}{1-\nu}\right)\xi J_0(\xi r) + \frac{1}{r}J_1(\xi r)\right]d\xi,$$

$$\sigma_{rz}^c(r, 0) \equiv 0. \tag{25}$$

We note the symmetry of the force-stress tensor and vanishing of the rotation and couple-stress components in the classical theory of elasticity.

4. Particular cases

We consider the following particular cases of the body-force and the body-couple.

Case (i) Conservative body-force, Case (ii) Body-couple as the curl of a vector function, Case (iii) Concentrated body-force, Case (iv) Concentrated body-couple.

The case where both the body-forces and the body-couples are prescribed can be obtained by superposing the corresponding expressions for the body-forces and the body-couples cases.

Case (i) Conservative body-force

Let us assume that there exists a scalar function $\Phi'(r, z)$ such that the components, X_r and X_z, of the body-force X are given by

$$(X_r, 0, X_z) = \left(-\frac{\partial \Phi'}{\partial r}, 0, -\frac{\partial \Phi'}{\partial z}\right) \tag{26}$$

such that

$$\Phi'(r, 0) \equiv 0.$$

Of course, in this case, we have

$$Y_\varphi(r, z) \equiv 0.$$

Hankel-transformation of (26) followed by its Laplace-transformation will yield

$$\bar{\bar{X}}_r(\xi, p) = \xi \bar{\bar{\Phi}}_0(\xi, p), \quad \bar{\bar{X}}_z(\xi, p) = -p\bar{\bar{\Phi}}_0(\xi, p), \tag{27}$$

where $\bar{\Phi}'_0$ denotes the zero-order Hankel-transform of the function Φ'. Hence, from (18) and (20), we get

$$\bar{M}_4(\xi) = -\bar{M}_5(\xi) = -2\mu\rho\zeta\bar{\bar{\Phi}}'_0(\xi, \xi), \quad \bar{M}_6(\xi) = -2\mu\rho\xi(\zeta - \xi)\bar{\bar{\Phi}}'_0(\xi, \xi).$$

So that at the stress-free boundary, we have

$$u_z(r, 0) = -\left(\frac{1-2\nu}{\mu}\right)\rho\int_0^\infty \zeta\xi\bar{\bar{\Phi}}'_0(\xi, \xi)\bar{\Delta}(\xi)J_0(\xi r)\,d\xi,$$

$$u_r(r, 0) = \left(\frac{1-2\nu}{\mu}\right)\rho\int_0^\infty \zeta\xi\bar{\bar{\Phi}}'_0(\xi, \xi)\bar{\Delta}(\xi)J_1(\xi r)\,d\xi,$$

$$w_\varphi(r, 0) = -\left(\frac{1-2\nu}{\mu}\right)\rho\int_0^\infty \xi^2(\zeta - \xi)\bar{\bar{\Phi}}'_0(\xi, \xi)\bar{\Delta}(\xi)J_1(\xi r)\,d\xi,$$

$$\sigma_{rr}(r, 0) = 2(1-2\nu)\rho\int_0^\infty \zeta\xi\bar{\bar{\Phi}}'_0(\xi, \xi)\bar{\Delta}(\xi)\left[\frac{\xi}{1-\nu}J_0(\xi r) - \frac{1}{r}J_1(\xi r)\right]d\xi,$$

$$\sigma_{\varphi\varphi}(r, 0) = 2(1-2\nu)\rho \int_0^\infty \zeta\xi\bar{\bar{\Phi}}_0'(\xi, \xi)\bar{\Delta}(\xi)\left[\left(\frac{\nu}{1-\nu}\right)J_0(\xi r) + \frac{1}{r}J_1(\xi r)\right]d\xi,$$

$$\sigma_{rz}(r, 0) = 2\rho(1-2\nu)k^2 \int_0^\infty \xi^3\bar{\bar{\Phi}}_0'(\xi, \xi)\bar{\Delta}(\xi)J_1(\xi r)d\xi, \tag{28}$$

and the couple-stress components $\mu_{r\varphi}$, $\mu_{\varphi r}$ are the same as given in (23),

Case (*ii*) *Body-couple as the curl of a vector function*

Let us assume that the body-couple $Y(r, z)$ is the curl of a vector function $\boldsymbol{\Psi}(r, z) = (\Psi_r, 0, \Psi_z)$ so that

$$Y_\varphi = \frac{\partial \Psi_r}{\partial z} - \frac{\partial \Psi_z}{\partial r},$$

and

$$\Psi_r(r, 0) \equiv \Psi_z(r, 0) \equiv 0. \tag{29}$$

Also, we have the body-forces vanishing throughout the medium. In this case, we obtain

$$\bar{\bar{Y}}_\varphi(\xi, p) = p\bar{\bar{\Psi}}_r(\xi, p) + \xi\bar{\bar{\Psi}}_z(\xi, p), \tag{30}$$

where $\bar{\bar{\Psi}}_r$ and $\bar{\bar{\Psi}}_z$ respectively are the first order and zero order Hankel-transforms of Ψ_r and Ψ_z.

Now from (18) and (20), we find that for this case $\overline{M}_4$, $\overline{M}_5$ and $\overline{M}_6$ are given by:

$$\overline{M}_4 = -\overline{M}_5 = (\lambda + 2\mu)J\xi\zeta\left[\bar{\bar{\psi}}_r(\xi, \xi) + \bar{\bar{\psi}}_z(\xi, \xi) - \bar{\bar{\psi}}_r(\xi, \zeta) - \frac{\xi}{\zeta}\bar{\bar{\psi}}_z(\xi, \zeta)\right]$$

$$\overline{M}_6 = (\lambda + 2\mu)J\xi^2(\zeta - \xi)[\bar{\bar{\psi}}_r(\xi, \xi) + \bar{\bar{\psi}}_z(\xi, \xi)] - \frac{\lambda + \mu}{2}J[\zeta\bar{\bar{\psi}}_r(\xi, \zeta) + \xi\bar{\bar{\psi}}_z(\xi, \zeta)]. \tag{31}$$

The expressions for the displacements, rotation and the required components of stresses and couple-stresses at the boundary $z = 0$ are then furnished by (21) and (23) by utilizing (31).

Case (*iii*) *Concentrated force*

Let us assume that the body-force $X = (X_r, 0, X_z)$ is concentrated at a point $(0, 0, h)$, $h > 0$, and acts in the direction $(0, 0, -1)$. Let its magnitude be P, so that we have

$$X_r \equiv \bar{\bar{X}}_r \equiv 0, \quad Y_\varphi \equiv \bar{\bar{Y}}_\varphi \equiv 0, \tag{32}$$

and

$$\rho X_z = -\frac{P\delta(r)\delta(z-h)}{2\pi r},$$

where δ denotes the Dirac delta function.

This implies that

$$\rho\bar{\bar{X}}_z(\xi, p) = -\frac{P}{2\pi}e^{-ph}, \quad h > 0. \tag{33}$$

Now from (32), (33), (18) and (20), we obtain

$$\bar{M}_4(\xi) = -\left(\frac{\mu}{1-2\nu}\right)\frac{P}{2\pi}\left\{[2(1-\nu)+\xi h]\frac{\zeta}{\xi}\mathrm{e}^{-\xi h} - 2(1-\nu)l^2\xi^2(\mathrm{e}^{-\xi h}-\mathrm{e}^{-\zeta h})\right\},$$

$$\bar{M}_5(\xi) = \left(\frac{\mu}{1-2\nu}\right)\frac{P}{2\pi}\left\{(1-2\nu+\xi h)\frac{\zeta}{\xi}\mathrm{e}^{-\xi h} + 2(1-\nu)l^2\xi(\xi\mathrm{e}^{-\zeta h}-\zeta\mathrm{e}^{-\xi h})\right\},$$

$$\bar{M}_6(\xi) = -\left(\frac{\mu}{1-2\nu}\right)\frac{P}{2\pi}\{[2(1-\nu)+\xi h](\zeta-\xi)\mathrm{e}^{-\xi h} + \xi(\mathrm{e}^{-\xi h}-\mathrm{e}^{-\zeta h})\}. \tag{33}$$

In this case, the displacements, rotation and stresses at the stress-free boundary then become:

$$u_r(r,0) = \frac{P}{4\pi\mu}\int_0^\infty \{(1-2\nu+\xi h)\zeta\mathrm{e}^{-\xi h} + 2(1-\nu)l^2\xi^2(\xi\mathrm{e}^{-\zeta h}-\zeta\mathrm{e}^{-\xi h})\}\bar{\Delta}(\xi)J_1(\xi r)\,d\xi,$$

$$u_z(r,0) = -\frac{P}{4\pi\mu}\int_0^\infty \{[2(1-\nu)+\xi h]\zeta\mathrm{e}^{-\xi h} - 2(1-\nu)l^2\xi^3(\mathrm{e}^{-\xi h}-\mathrm{e}^{-\zeta h})\}\bar{\Delta}(\xi)J_0(\xi r)\,d\xi,$$

$$w_\varphi(r,0) = -\frac{P}{4\pi\mu}\int_0^\infty \{[2(1-\nu)+\xi h](\zeta-\xi)\mathrm{e}^{-\xi h} + \xi(\mathrm{e}^{-\xi h}-\mathrm{e}^{-\zeta h})\}\,\xi\bar{\Delta}(\xi)J_1(\xi r)\,d\xi,$$

$$\sigma_{rr}(r,0) = \frac{P}{2\pi}\int_0^\infty [(1-2\nu+\xi h)\zeta\mathrm{e}^{-\xi h} + 2(1-\nu)l^2\xi^2(\xi\mathrm{e}^{-\zeta h}-\zeta\mathrm{e}^{-\xi h})]$$
$$\times\left[\left(\frac{\xi}{1-\nu}\right)J_0(\xi r) - \frac{1}{r}J_1(\xi r)\right]\bar{\Delta}(\xi)\,d\xi,$$

$$\sigma_{\varphi\varphi}(r,0) = \frac{P}{2\pi}\int_0^\infty [(1-2\nu+\xi h)\zeta\mathrm{e}^{-\xi h} + 2(1-\nu)l^2\xi^2(\xi\mathrm{e}^{-\zeta h}-\zeta\mathrm{e}^{-\xi h})]$$
$$\times\left[\left(\frac{\nu}{1-\nu}\right)\xi J_0(\xi r) + \frac{1}{r}J_1(\xi r)\right]\bar{\Delta}(\xi)\,d\xi,$$

$$\sigma_{rz}(r,0) = 2\mu k^2\left\{w_\varphi(r,0) + \frac{P}{4\pi\mu}\int_0^\infty [(2-2\nu+\xi h)\zeta\mathrm{e}^{-\xi h} - 2(1-\nu)l^2\xi^3(\mathrm{e}^{-\xi h}-\mathrm{e}^{-\zeta h})]\right.$$
$$\left.\xi\bar{\Delta}(\xi)J_1(\xi r)\,d\xi\right\},$$

$$\mu_{r\varphi}(r,0) = 2\mu l^2\left(\frac{\partial}{\partial r} - \frac{\gamma-\varepsilon}{2\mu l^2}\cdot\frac{1}{r}\right)w_\varphi(r,0),$$

$$\mu_{\varphi r}(r),0 = (\gamma-\varepsilon)\left(\frac{\partial}{\partial r} - \frac{2\mu l^2}{\gamma-\varepsilon}\cdot\frac{1}{r}\right)w_\varphi(r,0). \tag{34}$$

Now we will consider some limiting cases:

(a) When $h \to 0$; i.e., when the concentrated force is applied at the origin $r = z = 0$, we find

$$u_r^0(r,0) = \frac{P}{4\pi\mu}\int_0^\infty \{2(1-\nu)\xi\bar{A}(\xi) - 1)\}J_1(\xi r)\,d\xi,$$

$$u_z^0(r,0) = -\frac{P(1-\nu)}{2\pi\mu}\int_0^\infty \zeta\bar{A}(\xi)J_0(\xi r)\,d\xi,$$

$$w_\phi^0(r,0) = -\frac{P(1-\nu)}{2\pi\mu}\int_0^\infty \xi(\zeta-\xi)\bar{A}(\xi)J_1(\xi r)\,d\xi,$$

$$\sigma_{rr}^0(r,0) = \frac{P}{2\pi}\int_0^\infty \{2(1-\nu)\zeta\bar{A}(\xi) - 1\}\left\{\frac{\xi}{1-\nu}J_0(\xi r) - \frac{1}{r}J_1(\xi r)\right\}d\xi,$$

$$\sigma_{\phi\phi}^0(r,0) = \frac{P}{2\pi}\int_0^\infty \{2(1-\nu)\zeta\bar{A}(\xi) - 1\}\left\{\frac{\nu\xi}{1-\nu}J_0(\xi r) + \frac{1}{r}J_1(\xi r)\right\}d\xi,$$

$$\sigma_{rz}^0(r,0) = 2\mu r^2\left\{w_\phi^0(r,0) + \frac{P(1-\nu)}{2\pi\mu}\int_0^\infty \zeta\xi\bar{A}(\xi)J_1(\xi r)\,d\xi\right\},$$

$$u_{r\phi}^0(r,0) = 2\mu l^2\left(\frac{\partial}{\partial r} - \frac{\gamma-\varepsilon}{2\mu l^2}\cdot\frac{1}{r}\right)w_\phi^0(r,0),$$

$$\mu_{\phi r}^0(r,0) = \frac{\gamma-\varepsilon}{2\mu l^2}\,\mu_{r\phi}^0(r,0). \tag{35}$$

(b) When $r \to 0$; we find that the only non-vanishing quantities at the boundary $z = 0$, are given by

$$u_z(0,0) = \frac{P}{4\pi\mu}\int_0^\infty \{[2(1-\nu) + \xi h]\zeta e^{-\xi h} - 2(1-\nu)l^2\xi^3(e^{-\xi h} - e^{-\xi h})\}\bar{A}(\xi)\,d\xi,$$

$$\sigma_{rr}(0,0) = \sigma_{\phi\phi}(0,0)$$

$$= \frac{P}{4\pi}\frac{1+\nu}{1-\nu}\int_0^\infty \{(1-2\nu+\xi h)\zeta e^{-\xi h} + 2(1-\nu)l^2\xi^2(\xi e^{-\zeta h} - \zeta e^{-\xi h})\}\bar{A}(\xi)\,d\xi. \tag{36}$$

(c) When $r \to 0$, $h \to 0$; we find that

$$u_z^0(0,0) = -\frac{P(1-\nu)}{2\pi\mu}\int_0^\infty \zeta\bar{A}(\xi)\,d\xi,$$

$$\sigma_{rr}^0(0,0) = \sigma_{\phi\phi}^0(0,0) = \frac{P}{4\pi}\frac{1+\nu}{1-\nu}\int_0^\infty \{2(1-\nu)\zeta\bar{A}(\xi) - 1\}\xi\,d\xi, \tag{37}$$

and all other quantities vanish at the boundary face $z = 0$.

(d) When $h \to \infty$; all the displacements, rotations and stresses vanish at the face $z = 0$. This is to be expected, since we are assuming these quantities to vanish at an infinite distance from the plane $z = 0$ when the force is applied at a finite distance from this face. When the force is applied at an infinite distance from the face $z = 0$, then all the quantities must vanish at this face.

Case (iv) Concentrated couple

We assume that the body-couples acting in the interior of the half-space $z \geqq 0$ are equivalent to a single couple concentrated at a point $(0, 0, h)$, $h > 0$, whose moment, $(0, M, 0)$, is such as to keep all points lying on any given meridian plane before deformation on the same plane after deformation.

So, we have

$$X_r(r, z) \equiv X_z(r, z) \equiv 0,$$

and

$$JY_\varphi(r, z) = \frac{M\delta'(r)\delta(z-h)}{2\pi r}, \quad h > 0, \tag{38}$$

where $\delta'(r)$ is the derivative of the Dirac delta function $\delta(r)$ defined as

$$\delta'(r) = \lim_{\varepsilon \to 0} \frac{\delta(r+\varepsilon) - \delta(r-\varepsilon)}{2\varepsilon},$$

provided such a limit exists.

Hence, as before, we get

$$\bar{\bar{X}}_r \equiv \bar{\bar{X}}_z \equiv 0, \quad J\bar{\bar{Y}}_\varphi(\xi, p) = -\left(\frac{M\xi}{4\pi}\right)e^{-ph},$$

$$\bar{M}_4(\xi) = -\bar{M}_5(\xi) = 2\mu\left(\frac{1-\nu}{1-2\nu}\right)\frac{M\xi}{4\pi}(\xi e^{-\zeta h} - \zeta e^{-\xi h}),$$

$$\bar{M}_6(\xi) = -\left(\frac{\mu}{1-2\nu}\right)\frac{M\xi}{4\pi l^2}[2(1-\nu)l^2\xi(\zeta-\xi)e^{-\xi h} + e^{-\zeta h}], \tag{39}$$

which, when used in the general solution given by (21) and (23), yield the following demation and stress field at the stress-free boundary:

$$u_z(r, 0) = -\frac{M}{4\pi\mu}(1-\nu)\int_0^\infty \xi^2(\zeta e^{-\xi h} - \xi e^{-\zeta h})\bar{\Delta}(\xi)J_0(\xi r)\,d\xi,$$

$$u_r(r, 0) = \frac{M}{4\pi\mu}(1-\nu)\int_0^\infty \xi^2(\zeta e^{-\xi h} - \xi e^{-\zeta h})\bar{\Delta}(\xi)J_1(\xi r)\,d\xi,$$

$$w_\varphi(r, 0) = -\frac{M}{8\pi\mu l^2}\int_0^\infty \xi^2[2(1-\nu)l^2\xi(\zeta-\xi) + e^{-\zeta h}]\bar{\Delta}(\xi)J_1(\xi r)\,d\xi,$$

$$\sigma_{rr}(r, 0) = \frac{M}{2\pi}\int_0^\infty \xi^2(\zeta e^{-\xi h} - \xi e^{-\zeta h})\left[\xi J_0(\xi r) - \left(\frac{1-\nu}{r}\right)J_1(\xi r)\right]\bar{\Delta}(\xi)\,d\xi,$$

$$\sigma_{\varphi\varphi}(r, 0) = \frac{M}{2\pi}\int_0^\infty \xi^2(\zeta e^{-\xi h} - \xi e^{-\zeta h})\left[\nu\xi J_0(\xi r) + \left(\frac{1-\nu}{r}\right)J_1(\xi r)\right]\bar{\Delta}(\xi)\,d\xi,$$

$$\sigma_{rz}(r, 0) = 2\mu k^2\left[w_\varphi(r, 0) + (1-\nu)\frac{M}{4\pi\mu}\int_0^\infty \xi^3(\zeta e^{-\xi h} - \xi e^{-\zeta h})\bar{\Delta}(\xi)J_1(\xi r)\,d\xi\right],$$

$$\mu_{r\varphi}(r, 0) = 2\mu l^2\left(\frac{\partial}{\partial r} - \frac{\gamma - \varepsilon}{2\mu l^2}\cdot\frac{1}{r}\right)w_\varphi(r, 0),$$

$$\mu_{\varphi r}(r, 0) = (\gamma - \varepsilon)\left(\frac{\partial}{\partial r} - \frac{2\mu l^2}{\gamma - \varepsilon}\cdot\frac{1}{r}\right)w_\varphi(r, 0),$$

$$\mu_{\varphi z}(r, 0) = \mu_{z\varphi}(r, 0) = 0. \tag{40}$$

In this case also we will consider the following limiting cases:

(a) When $h \to 0$; we find

$$u_z^0(r, 0) = -\frac{M}{4\pi\mu}(1 - v)\int_0^\infty \xi^2(\zeta - \xi)\bar{\Delta}(\xi)J_0(\xi r)\,d\xi,$$

$$u_r^0(r, 0) = \frac{M}{4\pi\mu}(1 - v)\int_0^\infty \xi^2(\zeta - \xi)\bar{\Delta}(\xi)J_1(\xi r)\,d\xi,$$

$$w_\phi^0(r, 0) = -\frac{M}{8\pi\mu l^2}\int_0^\infty \xi^2[1 + 2(1 - v)l^2\xi(\zeta - \xi)]\bar{\Delta}(\xi)J_1(\xi r)\,d\xi,$$

$$\sigma_{rr}^0(r, 0) = \frac{M}{2\pi}\int_0^\infty \xi^2(\zeta - \xi)\left[\xi J_0(\xi r) - \frac{1 - v}{r}J_1(\xi r)\right]\bar{\Delta}(\xi)\,d\xi,$$

$$\sigma_{\phi\phi}^0(r, 0) = \frac{M}{2\pi}\int_0^\infty \xi^2(\zeta - \xi)\left[v\xi J_0(\xi r) + \frac{1 - v}{r}J_1(\xi r)\right]\bar{\Delta}(\xi)\,d\xi,$$

$$\sigma_{rz}^0(r, 0) = 2\mu k^2\left[w_\phi^0(r, 0) + \frac{M}{4\pi\mu}(1 - v)\int_0^\infty \xi^3(\zeta - \xi)\bar{\Delta}(\xi)J_1(\xi r)\,d\xi\right], \tag{41}$$

and $\mu_{r\phi}^0(r, 0)$ and $\mu_{\phi r}^0(r, 0)$ are given in terms of w_ϕ^0 by equations (35), where w_ϕ^0 is given above.

(b) When $r \to 0$; we find

$$u_z(0, 0) = -\frac{M}{4\pi\mu}(1 - v)\int_0^\infty \xi^2(\zeta e^{-\xi h} - \xi e^{-\zeta h})\bar{\Delta}(\xi)\,d\xi,$$

$$\sigma_{rr}(0, 0) = \sigma_{\phi\phi}(0, 0) = \frac{M}{4\pi}(1 + v)\int_0^\infty \xi^2(\zeta e^{-\xi h} - \xi e^{-\zeta h})\bar{\Delta}(\xi)\,d\xi, \tag{42}$$

while all other quantities vanish at $z = 0$.

(c) When $r \to 0$, $h \to 0$; we find

$$u_z^0(0, 0) = -\frac{M}{4\pi\mu}(1 - v)\int_0^\infty \xi^2(\zeta - \xi)\bar{\Delta}(\xi)\,d\xi,$$

$$\sigma_{rr}^0(0, 0) = \sigma_{\phi\phi}^0(0, 0) = -\frac{1 - v}{1 + v}\mu u_z^0(0, 0), \tag{43}$$

and all other quantities are zero.

5. Numerical results and general comments

The integrals occurring in this paper are difficult to evaluate analytically. Instead, we approximate them numerically.

These integrals, in general, are of the form

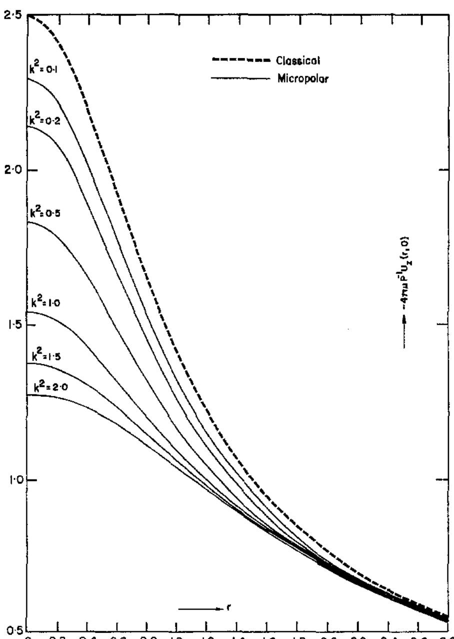
$$y = \int_0^\infty e^{-x} f(x)\, dx.$$

To compute this integral, Gaussian–Laguerre quadrature formulas are used to get the approximations:

$$y_n = \sum_{k=1}^{n} [A_k^{(n)} \cdot f(x_k)^{(n)}], \qquad n = 2, 3, \ldots,$$

which are exact whenever $f(x)$ is a polynomial upto the degree $2n - 1$. As pointed out by Krylov [11] the nodes $x_k^{(n)}$ are the roots of the Laguerre polynomials $L_n(x)$ of degree n. Values of the nodes x_k and the coefficients A_k for $n = 1(1)16(4)32$ are listed in the above-mentioned reference. In our computations, we have taken n to be 24 or 32.

Numerical computations has been done for values of r lying between 0 and 3, with the elastic constants ν and l^2 given by $\nu = 0.25$ and $l^2 = 1$. To make a comparative study, we let the micropolar constant α to vary between zero and infinity, and then observe the variation of the stress and the displacement fields with α. Since k^2 is given by $k^2 = 2\alpha/(\mu + \alpha)$,



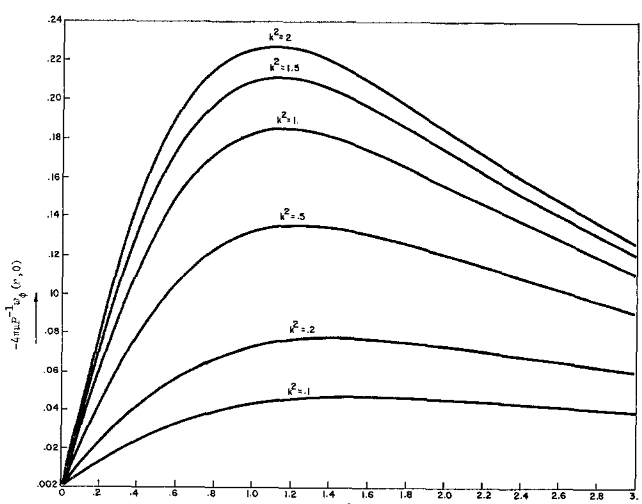


Figure 1. Variation of the normal displacement with r and k^2 on the stress-free boundary of the half-space under the action of concentrated force for $\nu = 0.25$, $l^2 = 1$ and $h = 1$.

Figure 2. Variation of the rotational component w_ϕ with r and k^2 at the stress-free boundary of the half-space under the action of concentrated force for $\nu = 0.25$, $l^2 = 1$ and $h = 1$.

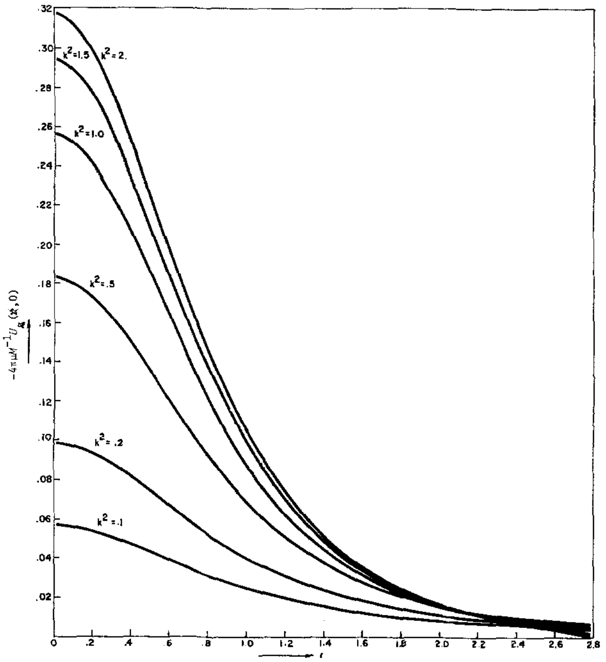


Figure 3. Variation of the normal displacement with r and k^2 at the stress-free boundary of the half-space under the action of concentrated couple for $\nu = 0$, $l^2 = 1$ and $h = 1$.

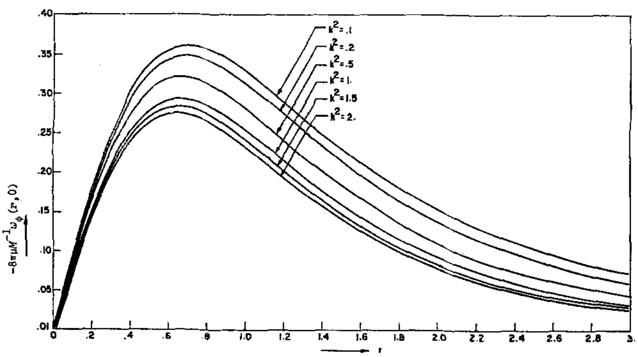


Figure 4. Variation of the rotational component w_ϕ with r and k^2 at the stress-free boundary of the half-space under the action of concentrated couple for $\nu = 0.25$, $l^2 = 1$ and $h = 1$.

we find that the interval $0 \leqq \alpha \leqq \infty$ for α is reduced to the interval $0 \leqq k^2 \leqq 2$ for k^2. So, we consider the values of k^2 given by 0, .1, .2, .4, .5, 1, 1.5 and 2. This includes the special case of the classical theory of elasticity given by $\alpha \to 0$ (i.e., $k^2 \to 0$).

Figs. 1–4 show the variation of the normal displacement and the rotation on the stress-free boundary of the half-space which is under the action of concentrated force and concentrated couple, considered separately, at the point (0, 0, 1) of the micropolar semi-infinite medium.

In general, we observe a marked difference between the classical and micropolar solutions for small values of r. It is interesting to note that this difference becomes significantly larger in absolute values with the increasing values of the micropolar constant α (Fig. 1). We also observe that the rotation is smaller in magnitude than the normal displacement. We find that the displacements converge to zero more rapidly than those observed in the case of the rotational component; and note a distinct behaviour of these quantities for the maximum value of the micropolar constant α as demonstrated by $k^2 = 2$ in these graphs. We note the contrasting behavior of displacements and rotation which, when considered separately for concentrated force and concentrated couple, show opposite trend in its values obtained for increasing micropolar constant α. We find that the normal displacement decreases in magnitude with α in the case of concentrated force while it increases in magnitude with α in the case of concentrated couple. The rotation components behave exactly in the opposite way.

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